MATH 102:107, CLASS 34 (MON NOV 27)

(1) (Warmup) Calculate!

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\quad$ $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ $\quad$ $\arccos(0) = \frac{\pi}{2}$ $\quad$ $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

(2) (Pulling a sled) You are pulling a sled with a rope. When the angle the rope makes with the ground is θ, the amount of force required to pull the sled is

$$F(\theta) = \frac{50}{\cos(\theta) + \mu\sin(\theta)}$$

where μ is a constant (μ is the coefficient of friction). In terms of μ, what value of θ minimizes this amount of force? (Hint: Maximize the denominator.)

Solution: To maximize the denominator, we set its derivative, $-\sin(\theta) + \mu\cos(\theta)$, equal to zero

$$-\sin(\theta) + \mu\cos(\theta) = 0 \implies \mu = \tan(\theta) \implies \boxed{\theta = \arctan(\mu)}$$

Now suppose that $\mu = 1$. Find the amplitude A and phase P such that $\cos(\theta) + \sin(\theta) = A\cos(\theta + P)$.

Solution: When $\mu = 1$, the maximum value occurs at $\theta = \arctan(1) = \frac{\pi}{4}$. Since an ordinary cosine curve has its maximum at $\theta = 0$, it follows that the phase is $P = -\frac{\pi}{4}$ (it is negative because the shift is to the *left*). This maximum value is

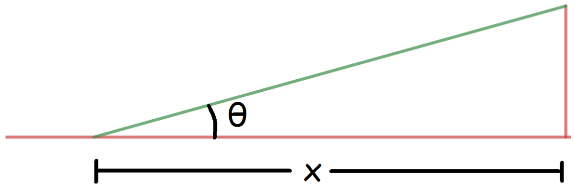
$$\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

and therefore the amplitude is $\sqrt{2}$. Thus,

$$\cos(\theta) + \sin(\theta) = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$$

Note: For a general value of μ, the phase will be $-\arctan(\mu)$ and the amplitude will be $\sqrt{1+\mu^2}$.

(3) (Viewing Angle) A building is 100m tall. You walk away from it at a constant speed of 1 m/s. The *viewing angle* is the angle formed between your line of sight to the top and your line of sight to the bottom (assume for simplicity that you are 0m tall). When you are a distance of x away from the base, how quickly is the viewing angle, θ, changing?



Solution: The relation between θ and x is summed up in the equation

$$\tan(\theta) = \frac{100}{x}$$

We are told that $x' = 1$, and want to calculate θ' entirely in terms of x. So if we use the Chain Rule to take the derivative of both sides (with respect to time),

$$\sec^2(\theta) \cdot \theta' = -\frac{100}{x^2} \cdot x'$$

Because $x' = 1$, the right side equals $-\frac{100}{x^2}$. Therefore,

$$\theta' = -\frac{100}{x^2}\cos^2(\theta)$$

Using the Pythagorean Theorem on the right triangle in the diagram, $\cos(\theta) = \frac{x}{\sqrt{x^2+100^2}}$[1], so $-\frac{100}{x^2}\cos^2(\theta) = -\frac{100}{x^2} \cdot \frac{x^2}{x^2+100^2} = \boxed{-\frac{100}{x^2+100^2}}$.

Qualitatively, this means that as your distance x gets larger, θ decreases (because θ' is negative), but it decreases at a slower and slower rate (because θ' is becoming smaller in size).

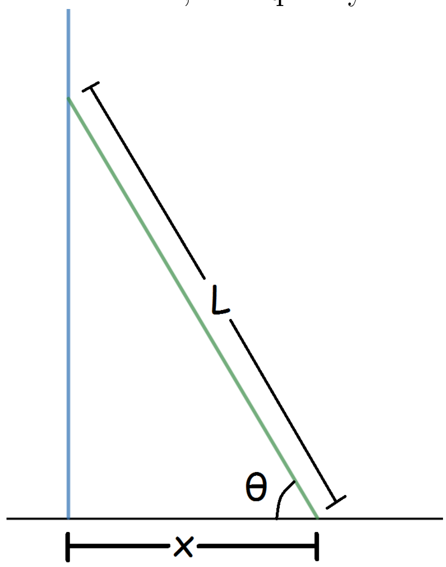
[1]This technique of using the Pythagorean theorem allows us to calculate identities such as

$$\cos(\arctan(x)) = \frac{x}{\sqrt{x^2+1}} \qquad \sin(\arctan(x)) = \frac{1}{\sqrt{x^2+1}} \qquad \cos(\arcsin(x)) = \sqrt{1-x^2}$$

With this technique, we can calculate the derivatives of the inverse trig functions. For example, to calculate the derivative of arcsin,

$$y = \arcsin(x) \implies \sin(y) = x \implies \cos(y) = \frac{dx}{dy}$$
$$\implies \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}$$

(4) (Slipping Ladder) A ladder of length L meters has one end on the ground and the other up on a wall. The base of the ladder is pulled away from the wall at a constant rate of 1 m/s. When the foot of the ladder is distance x away from the foot of the wall, how quickly is the angle of the ladder, θ, changing?



Solution: The relationship between x and θ is

$$\cos(\theta) = \frac{x}{L}$$

Taking the derivative of both sides with respect to time,

$$-\sin(\theta) \cdot \theta' = \frac{1}{L} \cdot x'$$

Since $x' = 1$, we have

$$\theta' = -\frac{1}{L \sin(\theta)}$$

By the Pythagorean Theorem, the height of the end of the ladder (up on the wall) is $\sqrt{L^2 - x^2}$. So $\sin(\theta) = \frac{\sqrt{L^2-x^2}}{L}$, and so

$$\theta' = -\frac{1}{L \cdot \frac{\sqrt{L^2-x^2}}{L}} = \boxed{-\frac{1}{\sqrt{L^2 - x^2}}}$$

Qualitatively, this means that initially (when $x = 0$) the top of the ladder is coming down at a rate of $\frac{1}{L}$. And as x increases, the top of the ladder comes down faster and faster. The fact that $\frac{1}{\sqrt{L^2-x^2}} \to \infty$ as $x \to L$ is the reason why the ladder eventually slips away from the wall! It cannot fall fast enough to stay in contact with the wall, as that would require an acceleration faster than 9.8 m/s^2.